Stochastic resonance as dithering

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A direct correspondence is demonstrated between the phenomenon of ''stochastic resonance'' in static nonlinear systems and the dithering effect well known in the theory of digital waveform coding. It is argued that many static systems displaying stochastic resonance are forms of dithered quantizers, and that the existence or absence of stochastic resonance in such systems can be predicted from the effects of ''dither averaging'' upon their transfer characteristics. Also, results are introduced regarding stochastic resonance in certain nonlinear systems with memory (e.g., hysteretic systems).

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I. INTRODUCTION

The term ''stochastic resonance'' is associated with the unexpected increase in output signal-to-noise ratio observed in certain nonlinear static or dynamical systems as the noise level at the system input increases. This usage dates back to the 1981 work of Benzi *et al.*[1] in climatic dynamics. Later the effect was observed in a variety of physical and biological systems from ring lasers to neurons [2]. However, the important distinction between stochastic resonance in dynamical versus static nonlinear systems has only been recognized relatively recently, with Gammaitoni [3] being the first author to directly acknowledge the correspondence between stochastic resonance and the dithering effect in static systems, and that no true resonance *per se* exists in such systems. Recently, a thorough theoretical treatment of static nonlinear systems has been published by Chapeau-Blondeau [4].

Quantization operations are required wherever it is necessary to reduce the precision of data prior to storage or transmission, as upon analog-to-digital conversion or arithmetical rounding of digital signals. As early as 1962 [5], engineers working in the field of picture coding discovered that the addition of random noise or *dither* prior to gray-scale quantization could modify the statistical character of the attendant error (the difference between the input and output of the quantizing system), resulting in perceptually preferable outputs. In the 1970s this technique was adapted for speech coding [6]. In such schemes, the dither was subtracted after the quantization operation, a technique known as *subtractive dithering*. The first theoretical examination of this technique was undertaken by Schuchman in 1964 with refinements appearing sporadically until the present [7–9].

Nonsubtractive dithering schemes, in which the dither is not subtracted following quantization, are the subject of more recent scholarly interest. The first investigations of such systems were undertaken in the late 1970s and early 1980s [10], but were not published. The primary results regarding these systems were published in the mid 1980s by Vanderkooy and Lipshitz [11], who later published the first thoroughgoing mathematical treatments with Wannamaker [8,9,12,13]. A major theoretical treatment of the subject has also been published by Stockham and Gray [14].

We will now proceed to demonstrate that dithered quantizing systems display the behavior known as stochastic resonance, and that this phenomenon corresponds to the dithering effect as it is conventionally understood. We will not undertake to compare subtractive and nonsubtractive dithering schemes here (see [8,9,13]), but will restrict our discussion to nonsubtractive schemes as these are more directly analogous to the stochastic resonance systems discussed in the physical literature.

II. DITHERED QUANTIZING SYSTEMS

Figure 1 shows a typical nonsubtractively dithered quantizing system with *system input* x, additive *dither* ν, *quantizer input* $w=x+\nu$, *system output* $y=Q(w)$, and *total error* $\varepsilon=y-x$. We will assume a (multilevel) quantizer of the midtread variety with transfer characteristic

$$Q(w)=\Delta\left\lfloor \frac{w}{\Delta}+\frac{1}{2}\right\rfloor, \tag{1}$$

where the ''floor'' operator $\lfloor\ \rfloor$ returns the greatest integer less than or equal to its argument. (Other common quantizer characterisitics can be treated similarly.) The quantizer step size Δ is commonly referred to as a LSB (least significant bit), since a change in input signal level of one step width corresponds to a change in the LSB of binary coded output.

We assume that the dither signal is statistically independent of the input signal so that the conditional probability density function (PDF) of ν given x is simply $p_{\nu|x}(\nu,x)=p_\nu(\nu)$. Then the nth moment of the system output given the system input, $E[y^n|x]$, is given by

$$E[y^n|x](x)=\int_{-\infty}^{\infty}Q^n(x+\nu)p_\nu(\nu)d\nu=Q^n(x)*p_\nu(-x), \tag{2}$$

where $*$ denotes the convolution operation. In particular, the *dither-averaged transfer characteristic* [11] $E[y|x]$ allows

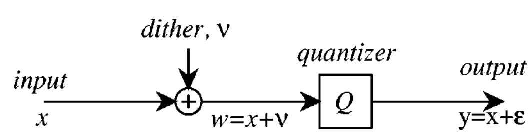

FIG. 1. Schematic of a typical nonsubtractively dithered quantizing system.

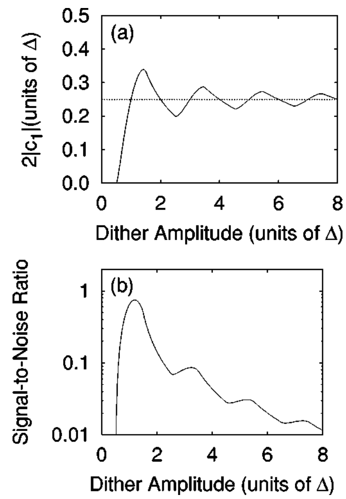

FIG. 2. (a) Peak amplitude of fundamental $2|\overline{c_1}|$, and (b) output signal-to-noise ratio as functions of normalized peak-to-peak dither amplitude γ for a nonsubtractively dithered quantizing system with sinusoidal input of peak amplitude $\Delta/4$.

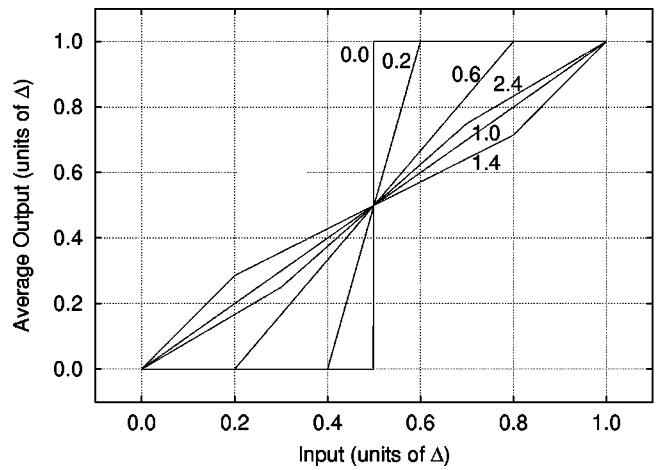

FIG. 3. $E[y|x](x), 0 \leqslant x \leqslant \Delta$, for various normalized peak-to-peak dither amplitudes γ in a nonsubtractively dithered quantizing system. Note that all lines display reflection symmetry through (0.5,0.5). The line for any nonzero integral value of γ coincides with the straight line shown for $\gamma=1.0$.

straightforward calculation of the expected value of the output as a function of time, $\overline{y}=E[y|x](x(t))$, when $x(t)$ is known.

If $x(t)$ is a periodic time function, then y is a cyclostationary stochastic process and $\overline{y(t)}$ is a periodic function of time which may be expanded in a Fourier series:

$$\overline{y(t)}=\sum_{n=-\infty}^{\infty}\overline{c_n}e^{j2\pi nt/T},$$

where T is the period of the input signal. The Fourier coefficients $\overline{c_n}$ are given by

$$\overline{c_n}=\frac{1}{T}\int_0^T\overline{y(t)}e^{-j2\pi nt/T}dt.$$

The power in the nth harmonic of $\overline{y}(t)$ is given by $2|\overline{c_n}|^2$. While the variance of y is a function of time, when $x(t)$ is periodic we may associate its average second moment over an integral number of periods (call it m_2) with the average output signal power.

Consider a sinusoidal input signal $x(t)=A\sin(2\pi t/T)$. As a simple measure of output signal-to-noise ratio (SNR) we use the power in the fundamental of $\overline{y(t)}$ relative to the rest of the power in the output signal:

$$\text{SNR}=\frac{2|\overline{c_1}|^2}{m_2-2|\overline{c_1}|^2}. \tag{3}$$

Let us consider uniformly distributed dithers; i.e., dithers with PDFs of the form

$$p_\nu(\nu)=\begin{cases}\dfrac{1}{\gamma\Delta}, & -\dfrac{\gamma\Delta}{2}<\nu\leqslant\dfrac{\gamma\Delta}{2}\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

For a sinusoidal input of peak amplitude $A=\Delta/4$, Fig. 2 plots both $2|\overline{c_1}|$ and output SNR as a function of normalized peak-to-peak dither amplitude γ. For $\gamma<\frac{1}{2}$, there is no output since the sum of the dither and signal is so small that it is invariably quantized to zero. As the dither amplitude increases, however, the SNR rises, attains a maximum value and subsequently decays in the manner characteristic of a system exhibiting stochastic resonance.

This result is qualitatively unchanged for any choice of continuous dither PDF, but the choice of a uniform PDF has special implications for this nonlinearity. In particular, we observe that, for integral values of γ, $2|\overline{c_1}|=\Delta/4=A$, so that $\overline{y(t)}=x(t)$. In fact, when this dither signal is used, $\overline{y(t)}=x(t)$ for *arbitrary* input signals [8,13]. This is demonstrated by Fig. 3, which shows $E[y|x](x)$, as computed from Eq. 2, for various values of γ. For $\gamma=1$ the dither-averaged transfer characteristic is a straight line through the origin with a slope of unity [11,3]. In this case, an arbitrary periodic signal is precisely recoverable from the system output by averaging over a large number of periods. We observe that for $\gamma=1.4$ the average input-output gain of the system is greater than unity for inputs smaller in magnitude than 0.5Δ, which explains how $2|\overline{c_1}|>A=0.25\Delta$ in the corresponding regime of Fig. 2(a). We conclude that the phenomenon of stochastic resonance in dithered quantizing systems is a direct consequence of the statistical modification of the transfer characteristic associated with dithering. As the characteristic approaches linearity, the proportion of the output signal which is coherent with the input increases, resulting in a SNR curve which rises until the increasing output noise level, associated with the increasing dither amplitude, causes the curve to fall once again.

A more thorough and sophisticated analysis of nonsubtractively dithered quantizers can be found in [8,12–14]. Many unexpected and important results can be proven for such systems, including formulas for moments, autocorrelation functions, and power spectra of the total error and system output. Here we mention only the very striking result that practical dither signals exist that render any desired mo-

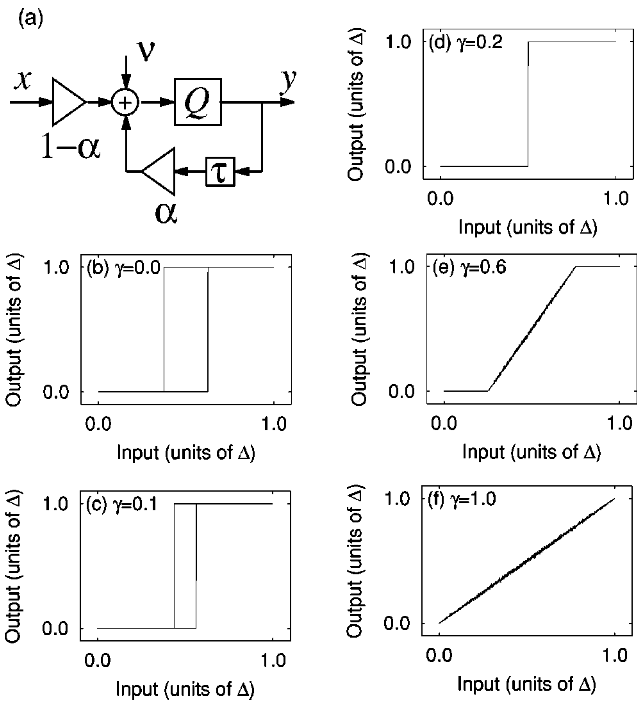

FIG. 4. (a) Hysteretic quantizer model and (b)–(f) dither-averaged transfer characteristics for quasistatic input signals computed by simulation using uniformly distributed dithers of various normalized peak-to-peak amplitudes γ (indicated at the upper left of each plot), with $\alpha=0.2$ and $0\leqslant x\leqslant\Delta$.

ments of the total error independent of the input signal. Such dither signals are now commonly used when processing digital audio signals.

III. HYSTERETIC QUANTIZERS

Figure 4(a) shows a schematized discrete-time model of a hysteretic quantizer that exhibits stochastic resonance. It consists of a quantizer Q in a feedback loop with a single-sample delay time τ of arbitrary size introduced to provide the memory necessary to simulate hysteresis. The model is appropriate for input signal bandwidths restricted to frequencies much less than $1/\tau$, at which frequencies the input-output gain of the circuit is unity. The width of the hysteresis loop is $[\alpha/(1-\alpha)]\Delta, 0\leqslant\alpha<1$ where Δ is the step width of the quantizer. Figure 4 also shows dither averaged transfer characteristics, generated through simulation, associated with this system for quasistatic inputs. We observe that the hysteresis loops disappear from the characteristic when the dither width exceeds the hysteretic width (i.e., when $\gamma\geqslant\alpha$) since no input value exists for which irreversible transitions in the output are possible. The complete linearization of the system when $\gamma=1$ is predictable, since we know from Sec. II above that the quantizer block behaves linearly whenever its input contains an independent dither component of this width. The requirement of independence implies that the dither noise must be independent and identically distributed (IID), since otherwise the dependence between successive dither values will appear via the feedback path as dependence between the dither and rest of the quantizer input.

We note that in a real physical system the option of adding the dither after the leading $(1-\alpha)$ gain reduction is unavailable. The topology shown is one of convenience for analytical purposes. Obviously, if the dither is added prior to the leading attenuation, then a dither width of $[1/(1-\alpha)]\Delta$ will be required in order to linearize the quantizer.

If the quantizer Q is replaced by a signum function (i.e., a comparator) of peak-to-peak height Δ, a Schmitt trigger or hysteretic threshold nonlinearity is modeled. The behavior of this system is similar to that shown in Fig. 4 (apart from a translation of the curves by $-\Delta/2$ along each of the axes in order to place the center of the quantizer's vertical step edge at the origin) as long as the magnitude of the input signal does not exceed $(2-\alpha-\gamma)/(1-\alpha)(\Delta/2)$, since in this case the nonlinearity can be regarded as a quantizer of which only a single step is being exercised.

IV. ARBITRARY STATIC NONLINEARITIES

Arbitrary static nonlinearities may also be analyzed using the approach of Sec. II. The nonlinearity Q of Eq. 1 can be replaced by an arbitrary function and the conditional output moments subsequently computed using Eq. 2. Conditional moments of the total error can then be computed by expanding $E[\varepsilon|x]=E\{[Q(x+\nu)-x)]^m|x\}$. Expressions for the system output and total error distributions can be computed whenever it is possible to characterize $p_{y|w}$ and $p_{\varepsilon|w}$, respectively. These, in turn, allow the computation of closed-form moment expressions. For instance, consider a threshold nonlinearity, $Q(w)=(\Delta/2)[1+\text{sgn}(w-\Omega)]$. By inspection,

$$p_{y|w}(y,w)=\tfrac{1}{2}[1-\text{sgn}(w-\Omega)]\delta(y)+\tfrac{1}{2}[1+\text{sgn}(w-\Omega)]\delta(y-\Delta),$$

so that

$$p_y(y)=\int_{-\infty}^{\infty}p_{y|w}(y,w)p_w(w)dw=F_w(\Omega)\delta(y)+[1-F_w(\Omega)]\delta(y-\Delta),$$

where F_w is the cumulative distribution function of w. Calculation of moment expressions is now straightforward and conditional moment expressions can be obtained by substituting $p_w(w)=p_{w|x}(w,x)=p_\nu(w-x)$:

$$E[y^m|x](x)=\Delta^m[1-F_\nu(\Omega-x)].$$

With a dither noise uniformly distributed according to Eq. 4, this becomes

$$E[y^m|x](x)=\Delta^m\times\begin{cases}0, & x<\Omega-\gamma\Delta/2\\ (1/\gamma\Delta)(x-\Omega)+\tfrac{1}{2}, & |x-\Omega|\leqslant\gamma\Delta/2\\ 1, & x>\Omega+\gamma\Delta/2.\end{cases}$$

Note that for $\gamma=1$ we have $\overline{y(t)}=x(t)-\Omega+\Delta/2$ so long as $|x(t)-\Omega|\leqslant\Delta/2$ for all t. This is hardly surprising insofar as the dithered nonlinearity may be regarded as a single step extracted from a (translated) dithered quantizer staircase under these conditions. Of course, similar computations can be performed for the total error signal if a detailed description of its distribution is desired.

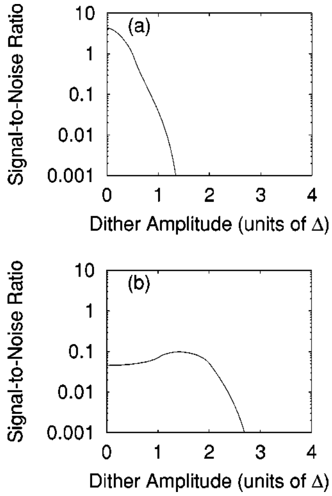

FIG. 5. Output signal-to-noise ratio as a function of normalized peak-to-peak dither amplitude γ for the nonlinearity of Eq. 5 with uniformly distributed dither and sinusoidal inputs of peak amplitudes (a) $\Delta/4$ and (b) Δ.

Although this approach is clearly useful from a practical standpoint, it is our experience that the many elegant theoretical results regarding dithered quantizers usually have no counterparts for other dithered nonlinearities (the use of characteristic functions [8,13] in particular being generally fruitless). It is the highly regular form of the quantizer nonlinearity that allows for so many surprising results.

V. PREDICTING STOCHASTIC RESONANCE

The question remains as to when stochastic resonance should be expected in a given static nonlinear system. The most important contributing factor seems to be the occurrence of maxima in the numerator of Eq. 3, these typically being amplified by attendant decreases in the denominator as the output signal power $2|\overline{c_1}|^2$ increases. Now, the convolution in Eq. 2 of the nonlinear transfer characteristic with a continuous dither PDF corresponds to a local weighted-average or smoothing of the characteristic, with larger dither amplitudes resulting in greater smoothing. An increase in output signal power is to be expected with increasing dither amplitude whenever the magnitude of the averaged transfer characteristic undergoes an attendant increase at those input values where the input signal resides for a significant amount of time. For instance, considered as an ensemble, the values of a sinusoid with peak amplitude A have a PDF of the form

$$p_x(x)=\begin{cases}1/(\pi\sqrt{A^2-x^2}), & |x|<A\\ 0 & \text{otherwise,}\end{cases}$$

which becomes large in the neighborhoods of $x=\pm A$. When an increase occurs in the magnitude of the dither-averaged transfer characteristic in these neighborhoods, a concomitant increase in output signal power is expected. This has already been observed in association with dithered quantizers in Sec. II. It is particularly well illustrated by the ''doublet''-like nonlinearity

$$Q(w)=\begin{cases}-1, & -\frac{1}{2}<w<0\\ 1, & 0<w<\frac{1}{2}\\ 0 & \text{otherwise,}\end{cases} \tag{5}$$

which displays stochastic resonance with sinusoidal inputs for certain input amplitudes only, revealing the complete inappropriateness of the term ''resonance'' for this phenomenon. In particular, smoothing of this characteristic near $x=0$ decreases its magnitude there, attenuating small inputs, while increasing its magnitude for $|x|>\Delta/2$, thereby amplifying large inputs (see Fig. 5.) In most instances it is similarly possible to predict the appearance of stochastic resonance in other static nonlinear systems by considering the effect upon input signals of dither averaging the nonlinear transfer characteristic.

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